Angular Momentum in Quantum Mechanics

- This section is best read in conjunction with the section on Orbital angular momentum in the previous chapter.
- Known: $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$, etc

  $[\hat{L}^2, \hat{L}_z] = 0$, etc.

Can find simultaneous eigenstates of $\hat{L}^2$ and one component $(\hat{L}_z)$

In spherical coordinates, $Y_{\ell m_\ell}(\theta, \phi)$ are simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$

H. General Solution to Angular Momentum Eigenvalue Problem

- Meaning: Rely only on $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, and $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ and their commutators

$$\left.\begin{aligned} [\hat{L}_x, \hat{L}_y] &= i\hbar \hat{L}_z \;; & [\hat{L}^2, \hat{L}_x] &= 0 \\ [\hat{L}_y, \hat{L}_z] &= i\hbar \hat{L}_x \;; & [\hat{L}^2, \hat{L}_y] &= 0 \\ [\hat{L}_z, \hat{L}_x] &= i\hbar \hat{L}_y \;; & [\hat{L}^2, \hat{L}_z] &= 0 \end{aligned}\right\} \quad (1)$$

But <u>not</u> on any representation (e.g. spherical coordinates)

Know: Can find simultaneous eigenstates of $\hat{L}^2$ and $\hat{L}_z$ (one component)

<u>Question</u>: What can be said about the eigenstates and eigenvalues of $\hat{L}_z$ and $\hat{L}^2$ based entirely on commutation relations?

- To stress that we are considering <u>general</u> angular momentum, but <u>not only</u> orbital angular momentum, we use new symbols $(\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}^2)$ for $(\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}^2)$

- <u>General Angular Momentum</u>

Eq. (1) [the commutation relations] <u>DEFINES</u> angular momentum in QM

<u>Meaning</u>: Whatever quantity that satisfies Eq. (1), it is <u>called</u> an angular momentum in QM

- Let ϕ be <u>a simultaneous eigenstate</u> of $\hat{J}^2$ and $\hat{J}_z$

$$\hat{J}^2 \phi = \alpha \phi \qquad \text{and} \qquad \hat{J}_z \phi = \beta \phi \tag{2}$$

eigenvalue of $\hat{J}^2$ eigenvalue of $\hat{J}_z$

$$\alpha \geq \beta^2 \tag{3}$$

[component <u>cannot</u> be longer than length of vector]

- Introducing $\hat{J}_+$ and $\hat{J}_-$

$$\underbrace{\hat{J}_+ \equiv \hat{J}_x + i\hat{J}_y}_{\text{Raising Operator}} \quad ; \quad \underbrace{\hat{J}_- \equiv \hat{J}_x - i\hat{J}_y}_{\text{Lowering Operator}} \tag{4}$$

[<u>Not</u> Hermitian] [<u>Not</u> Hermitian]

No problem! They do not represent physical quantities.

- $[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$ (5)

Why? $\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 - i \overbrace{[\hat{J}_x, \hat{J}_y]}^{i\hbar \hat{J}_z}$

$= \hat{J}_x^2 + \hat{J}_y^2 + \hbar \hat{J}_z$ (Useful later)

Similarly $\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z$ (Useful later)

$$[\hat{J}_+, \hat{J}_-] = \hat{J}_+ \hat{J}_- - \hat{J}_- \hat{J}_+ = 2\hbar \hat{J}_z$$ Done!

- $[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x + i\hat{J}_y] = [\hat{J}_z, \hat{J}_x] + i[\hat{J}_z, \hat{J}_y]$

$= i\hbar \hat{J}_y + i(-i\hbar \hat{J}_x)$

$= \hbar(\hat{J}_x + i\hat{J}_y) = \boxed{\hbar \hat{J}_+ = [\hat{J}_z, \hat{J}_+]}$ (6)

Similarly, $\boxed{[\hat{J}_z, \hat{J}_-] = -\hbar \hat{J}_-}$ (7)

▪ What do $\hat{J}_+$ and $\hat{J}_-$ do?

$$\hat{J}_z(\hat{J}_+\phi) = (\hat{J}_+\hat{J}_z + \hbar\hat{J}_+)\phi \qquad (\because [\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+)$$

$$= (\beta+\hbar)(\hat{J}_+\phi) \quad (8) \qquad (\because \hat{J}_z\phi = \beta\phi)$$

$$\hat{J}_z(\hat{J}_-\phi) = (\hat{J}_-\hat{J}_z - \hbar\hat{J}_-)\phi \qquad (\because [\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_-)$$

$$= (\beta-\hbar)(\hat{J}_-\phi) \quad (9) \qquad (\because \hat{J}_z\phi = \beta\phi)$$

Key Point

$\therefore$ If ϕ is eigenstate of $\hat{J}_z$ with eigenvalue β,

$(\hat{J}_+\phi)$ is also an eigenstate of $\hat{J}_z$ with eigenvalue $(\beta+\overbrace{\hbar}^{\text{one }\hbar\text{ higher}})$

$(\hat{J}_-\phi)$ is also an eigenstate of $\hat{J}_z$ with eigenvalue $(\beta-\underbrace{\hbar}_{\text{one }\hbar\text{ lower}})$ (10)

This is why $\hat{J}_+$ and $\hat{J}_-$ are called <u>raising</u> and <u>lowering</u> operators.

- $\hat{J}^2\phi = \alpha\phi$. Will $\hat{J}_+$ and $\hat{J}_-$ change α?

$$[\hat{J}^2, \hat{J}_\pm] = [\hat{J}^2, \hat{J}_x \pm i\hat{J}_y] = \underbrace{[\hat{J}^2, \hat{J}_x]}_{0} \pm i \underbrace{[\hat{J}^2, \hat{J}_y]}_{0} = 0$$

$$\hat{J}^2(\hat{J}_+\phi) = \hat{J}_+\hat{J}^2\phi = \alpha(\hat{J}_+\phi) \quad (11) \quad (\because \hat{J}^2\phi = \alpha\phi)$$

$$\hat{J}^2(\hat{J}_-\phi) = \hat{J}_-\hat{J}^2\phi = \alpha(\hat{J}_-\phi) \quad (12)$$

$$\boxed{\begin{array}{l}\therefore \text{ If } \phi \text{ is an eigenstate of } \hat{J}^2 \text{ with eigenvalue } \alpha, \\ (\hat{J}_+\phi) \text{ and } (\hat{J}_-\phi) \text{ are also eigenstates of } \hat{J}^2 \text{ with eigenvalue } \alpha\end{array}} \quad (13)$$

So, $\hat{J}_+$ and $\hat{J}_-$ respectively raise and lower eigenstates up or down the $\hat{J}_z$ eigenvalues in step of $\hbar$, but they don't change α.

- Recall: $\alpha \geq \beta^2$ [component can't be longer than full length]

  For a particular value of α, there must be a maximum value of β (say β_{max}) and a minimum value of β (say β_{min})

$$\beta_{max} \leftrightarrow \text{eigenstate } \phi_{\beta_{max}} \;;\; \beta_{min} \leftrightarrow \text{eigenstate } \phi_{\beta_{min}}$$

<u>Key ideas:</u>

$\hat{J}_+$ carries $\hat{J}_z$ eigenstate up in eigenvalue, but <u>there is a bound</u>

Eq.(8): $\hat{J}_z(\hat{J}_+\phi) = (\beta+\hbar)(\hat{J}_+\phi)$

- When we get to $\phi_{\beta_{max}}$, $\hat{J}_+\phi_{\beta_{max}}$ should <u>not</u> take it further higher
- But Eq.(8) must still remain true, i.e.

$$\boxed{\hat{J}_+\phi_{\beta_{max}} = 0} \quad (13)$$

{ defines $\phi_{\beta_{max}}$
{ gives an eq. for solving $\phi_{\beta_{max}}$.

$\hat{J}_-$ carries $\hat{J}_z$ eigenstate down in eigenvalue, but <u>there is a bound</u>

Eq.(9): $\hat{J}_z(\hat{J}_-\phi) = (\beta-\hbar)(\hat{J}_-\phi)$

- When we get to $\phi_{\beta_{min}}$, $\hat{J}_-\phi_{\beta_{min}}$ should <u>not</u> take it further lower
- But Eq.(9) must still remain true, i.e.

$$\boxed{\hat{J}_-\phi_{\beta_{min}} = 0} \qquad (14)$$

<u>Recall</u>:

$$\hat{J}_+\hat{J}_- = \hat{J}_x^2 + \hat{J}_y^2 + \hbar\hat{J}_z = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$$

$$\hat{J}_-\hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar\hat{J}_z = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$$

- $$\hat{J}_-\underbrace{\left(\hat{J}_+\phi_{\beta_{max}}\right)}_{0} = \left(\hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z\right)\phi_{\beta_{max}} = 0$$

$$\Rightarrow \left(\alpha - \beta_{max}^2 - \hbar\beta_{max}\right)\phi_{\beta_{max}} = 0$$

$$\Rightarrow \boxed{\alpha = \beta_{max}(\beta_{max} + \hbar)} \qquad (15)$$

- $$\hat{J}_+\underbrace{\left(\hat{J}_-\phi_{\beta_{min}}\right)}_{0} = \left(\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z\right)\phi_{\beta_{min}} = 0$$

$$\Rightarrow \left(\alpha - \beta_{min}^2 + \hbar\beta_{min}\right)\phi_{\beta_{min}} = 0$$

$$\Rightarrow \boxed{\alpha = \beta_{min}(\beta_{min} - \hbar)} \qquad (16)$$

- α is the same in Eqs. (15) and (16): <u>β_{max} and β_{min} are related by</u>

$$\boxed{\beta_{min} = -\beta_{max}} \qquad (17) \quad \text{Key Result}$$

$\therefore$ The smallest z-component is −[largest z-component]

<u>Remark</u>: (17) looks familiar! E.g. d states in <u>orbital</u> angular momentum

$Y_{22}, Y_{21}, Y_{20}, Y_{2,-1}, Y_{2,-2}$, i.e. $\beta_{max} = +2\hbar$, $\beta_{min} = -\beta_{max} = -2\hbar$.

But, we need not be considering orbital AM here!

- Saw that $\hat{J}_z$ eigenvalues differ in step of $\hbar$

  [$\hat{J}_-$ lowers eigenvalue from β_{max} to $(\beta_{max}-\hbar)$, $(\beta_{max}-2\hbar)$, ... β_{min}.]

$$\therefore \quad \beta_{max} - \beta_{min} = \text{integer} \times \hbar = n\hbar \quad [n \text{ is an integer}]$$

(non-negative)

But $\beta_{min} = -\beta_{max}$, $\quad \beta_{max} - (-\beta_{max}) = 2\beta_{max} = n\hbar$

$$\Rightarrow \boxed{\beta_{max} = \frac{n}{2}\hbar} \quad (18)$$

<u>Meaning</u>: Two cases
- n = even (e.g. $n=2$, J_z: $1\hbar, 0, -\hbar$)
- n = odd (e.g. $n=1$, J_z: $\frac{1}{2}\hbar, -\frac{1}{2}\hbar$)

- Write $j = \frac{n}{2}$ ($\because$ j is <u>either integer</u> OR <u>half-integer</u>)

$$\beta_{max} = j\hbar \ , \ \beta_{min} = -j\hbar \qquad [J_z = j\hbar, (j-1)\hbar, \cdots, -j\hbar]$$

Eq.(15): $\underset{\boxed{\text{eigenvalue of } \hat{J}^2}}{\alpha} = \beta_{max}(\beta_{max}+\hbar) = j\hbar(j\hbar+\hbar) = \underbrace{j(j+1)\hbar^2}_{\boxed{\text{must be of this form}}}$

<u>Conclusion</u>: Based on $\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}^2$ commutation relations only, $\hat{J}^2$ has eigenvalues of the form $j(j+1)\hbar^2$, with j taking on integers <u>OR</u> half-integers. (19)

For given value of j, $\hat{J}_z$ has eigenvalues that run from $j\hbar$ to $-j\hbar$ in steps of $\hbar$, giving $(2j+1)$ possible values.

Remarks

- The Most general statements about $\hat{J}^2$ and $\hat{J}_z$ eigenvalues
- Note: j can be half-integers OR integers

e.g. $j = 1/2$, $j = 3/2, \cdots$ [New]

e.g. Electron has Spin Angular Momentum with $j = 1/2$ (OR $s = 1/2$)

$j = 0$, $j = 1$, $j = 2, \cdots$ [Saw this in orbital AM, $\ell = 0$, $\ell = 1$, $\ell = 2$]

- Why does ℓ take on integers (but not half-integers) only?
  - Orbital angular momentum → θ, ϕ in real space

$$\Phi(\phi) = \Phi(\phi + 2\pi) \Rightarrow \text{z-components are } m_\ell \hbar$$

integer!

more than commutators only

- $j = 1/2$ case is particularly important (related to <u>spin</u>)

$$j = 1/2, \qquad J^2 = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 \quad \text{OR} \quad J = \sqrt{\frac{3}{4}}\,\hbar$$

$$J_z = \underbrace{\frac{\hbar}{2},\ -\frac{\hbar}{2}}_{2 \text{ possible values } \left[2\cdot\frac{1}{2}+1=2\right]}$$

- All particles in the Universe (e.g. those in Standard Model)
  - Spin Angular Momentum (j = integer) (Bosons)
  - Spin Angular Momentum (j = half-integer) (Fermions)

<u>Summary</u>: Writing the results in a formal way

- $\hat{J}_z$: Eigenvalues written as $m_j \hbar$ (introducing m_j)

  Thus, for given j, $m_j = j, j-1, \ldots, -j$

- A Simultaneous eigenstate of $\hat{J}^2$ and $\hat{J}_z$ is defined (or <u>labelled</u>) by j and m_j [j gives $j(j+1)\hbar^2$ for $\hat{J}^2$ and m_j gives $m_j\hbar$ for $\hat{J}_z$]

- Use the symbol $|j\, m_j\rangle$ for the simultaneous eigenstate

$$\therefore\ \hat{J}^2 |j\, m_j\rangle = j(j+1)\hbar^2\, |j\, m_j\rangle \tag{20}$$

$$\hat{J}_z |j\, m_j\rangle = m_j\hbar\, |j\, m_j\rangle$$

Given j, m_j goes from j to $-j$ in steps of 1

- Eq. (20) states the general results <u>without</u> referring to any way (<u>representation</u>) of expressing the AM operators AND the eigenstates
- In this notation, Eq. (10) says

$$\hat{J}_+ |j\ m_j\rangle \underset{\uparrow \text{proportional}}{\propto} |j\ m_j+1\rangle; \quad \hat{J}_- |j\ m_j\rangle \propto |j\ m_j-1\rangle$$

- See p. X-33, $\hat{J}_- \hat{J}_+ = \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z = \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z$

$$\hat{J}_- \hat{J}_+ |j\ m_j\rangle = (\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z)|j\ m_j\rangle = [j(j+1) - m_j(m_j+1)]\hbar^2 |j\ m_j\rangle$$

using this notation

- Eq. (20) works for all angular momenta